लक्ष्यं विज्ञानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 3rd Semester Examination, 2019

MTMACOR05T-MATHEMATICS (CC5)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5=10$

(a) Using ε - δ approach of limit, prove that $\lim\limits_{x\to0}x^2\sin\frac{1}{x}=0$.

(b) Show that $\lim\limits_{x\to\infty}\frac{[x]}{x}=1$ where $[x]$ denotes the greatest integer not greater than x.

(c) $f(x) \;\; := \begin{cases}-1\ , & \text{when } x \text{ is rational}\\ \ \ 1\ , & \text{when } x \text{ is irrational}\end{cases}$

Prove that $f(x)$ is discontinuous at every point of R.

(d) Is $f(x)=\frac{1}{x}$ uniformly continuous in (0, 1)? Give reasons in support of your answer.

(e) $f(x)=1-|x-1|$, $x\in[0, 2]$. Dose it satisfy the conditions of Rolle's theorem? Give reasons.

(f) Verify Cauchy's Mean Value Theorem for the functions $f(x)=e^x$ and $g(x)=e^{-x}$ in [0, 1].

(g) Find the extreme value of the function f in its domain: $f(x)=\dfrac{\log x}{x}$.

(h) Show that there exists a root of $x+x\log x-3=0$ in (1, 3).

2. (a) Let $f:D\longrightarrow\mathbb{R}$ be a function and a be a limit point of $D\subseteq\mathbb{R}$. Show that $\lim\limits_{x\to a}f(x)=l$, for some real number l if and only if for every sequence $\{x_n\}$ in D converging to a, the sequence $\{f(x_n)\}$ converges to l. 5

(b) Use Cauchy's general principle to prove that $\lim\limits_{x\to\infty}\cos x$ does not exist. 3

3. (a) Let $f:D\longrightarrow\mathbb{R}$ be continuous at $a\in D$, $D\subseteq\mathbb{R}$. Let $f(a)\neq0$. Show that there is a neighbourhood N of a so that f has the same sign as $f(a)$ in $N\cap D$. 4

(b) Let $f:[a, b] \longrightarrow \mathbb{R}$ be a continuous function so that $f(a)$ and $f(b)$ have opposite signs. Show that there is a $c \in (a, b)$ such that $f(c)=0$. 4

4. (a) Let $f: I \longrightarrow \mathbb{R}$ be a uniformly continuous function on an interval I of $\mathbb{R}$. Let $\{x_n\}$ be a Cauchy sequence in I. Show that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. 4

(b) Show that $f(x)=\dfrac{1}{1+x^2}$ is uniformly continuous on $(-\infty, \infty)$. 4

5. (a) Let $f: I \longrightarrow \mathbb{R}$ be a function where I is an interval in $\mathbb{R}$. Let f be differentiable at $x \in I$. Show that f is continuous at x. Give an example with proper justification to show that the converse is not true. 2+2

(b) Let $f: I \longrightarrow \mathbb{R}$ and $g: J \longrightarrow \mathbb{R}$ be two functions defined on intervals I, J of $\mathbb{R}$ so that Image $f \subseteq J$. Let f be differentiable at $c \in I$ and g be differentiable at $f(c) \in J$. Show that the composition $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c))f'(c)$. 4

6. (a) State and prove Darboux theorem on derivative. 1+4

(b) Show that the function $f:\ \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x)=\begin{cases} x^2 \sin\frac{1}{x^2} & , x \neq 0 \\ 0 & , x=0 \end{cases}$ is differentiable on $\mathbb{R}$ but $f'(x)$ is not continuous. 3

7. (a) State and prove Cauchy's Mean Value Theorem. 1+3

(b) If f is differentiable on $[0, 1]$, show by the above theorem that $f(1)-f(0)=\dfrac{f'(x)}{2x}$ has at least one solution in $(0, 1)$. 2

(c) Let I be an interval. If a function $f: I \longrightarrow \mathbb{R}$ be such that $f'(x)$ exists and is bounded on I, then show that f is uniformly continuous on I. 2

8. (a) State Rolle's theorem. 1

(b) Show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x = 1$. 3

(c) Apply mean value theorem to prove that $\dfrac{x}{1+x} < \log(1+x) < x$ if $x>0$. 4

9. (a) State and prove Taylor's theorem with Lagrange's form of remainder. 1+3

(b) Obtain Maclaurin's infinite series expansion of $(1+x)^n$, $|x|<1$, where $n \in R-N$. 4

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