WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2021

PHSACOR08T-PHYSICS (CC8)

MATHEMATICAL PHYSICS-III

Time Allotted: 2 Hours | Full Marks: 40

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Question No. 1 is compulsory and answer any *two* from the rest

1. Answer any ***ten*** questions from the following: $2\times10=20$

(a) If $i=\sqrt{-1}$, show that $i^{-2i}=e^{\pi}$.

(b) Show that $f(z)=|z|^2$ satisfies Cauchy-Riemann conditions only at $z=0$.

(c) Show that there are exactly *n* distinct *n*-th roots of z.

(d) Find the residues at the poles of the function $f(z)=\cdot\dfrac{z^2+4}{z^3+2z^2+2z}$.

(e) Evaluate $\oint_C (z^4+2z^3+3z^2+4z+5)dz$, where C is a unit circle around $z=0$.

(f) Give reasons whether the Fourier transformation of the following function exists.

$$f(x)=\begin{cases} 1, & \text{if } x \text{ is a rational number} \\ -1, & \text{if } x \text{ is a irrational number} \end{cases}$$

(g) If $F(k)$ is the Fourier transform of $f(x)$ and $G(k)$ is the Fourier transform of $g(x)=f(x+a)$, then show that $G(k)=e^{-iak}F(k)$.

(h) Find the Fourier transform of exponential decay function $e^{-\frac{t}{\tau}}$.

(i) Evaluate the Fourier transformation of $f(x)=(x^2-x+1)\delta(x-1)$. Where $\delta(x)$ is the Dirac-delta distribution.

(j) Show that the scalar potential associated with a vector field which is irrotational and also solenoidal, obeys Laplace's equation.

(k) If ϕ be a function of r only, then show that $\nabla^2\phi = \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr}$.

(l) Show that under similarity transformation the trace of a matrix remains invariant.

(m) If $\boldsymbol{A}$ is non-singular, then show that the eigenvalues of $\boldsymbol{A}^{-1}$ are the reciprocals of those of $\boldsymbol{A}$ and every eigen vectors of $\boldsymbol{A}$ is also an eigen vectors of $\boldsymbol{A}^{-1}$.

(n) Let $\boldsymbol{P}$ be a hermitian matrix with property $\boldsymbol{P}^2 = \boldsymbol{P}$. Show that for any vector $\mathbf{x}$, the vectors $\boldsymbol{P}\mathbf{x}$ and $(1-\boldsymbol{P})\mathbf{x}$ are orthogonal.

2. (a) Find the roots of the equation $z^4 = -256$ and plot the roots in the complex plane. 2

(b) Evaluate $I = \int_0^{2\pi} \frac{\cos 2\theta}{a^2 + b^2 - 2ab\cos\theta} d\theta,\ b > a > 0$. 4

(c) If the inner product between two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined by $(\boldsymbol{A}, \boldsymbol{B}) = \text{Trace}(\boldsymbol{A}^{\dagger}\boldsymbol{B})$, show that $\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ are orthogonal to each other. Determine the eigenvalues of the matrices $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. 2+2

3. (a) Find the Laurent series of $f(z) = \frac{1}{z(z-2)^3}$ about $z = 0$. 2

(b) For source free region, the electrostatic potential $V(r)$ can be obtained from the solution of Laplace equation, which reads $\nabla^2 V = 0$. A unit disk $(r \le 1)$ has no source of charge on it. The potential on the rim $(r = 1)$ is given by 2 sin 4θ, where θ is the polar angle. Obtain an expression for potential inside the disk. 4

(c) Verify Caley-Hamiltonian theorem for the matrix $\boldsymbol{A} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$, and hence find $\boldsymbol{A}^{-1}$. 4

4. (a) Find the Fourier transformation of $f(t) = e^{-|t|}$, and hence using inversion, deduce that $\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$. 3

(b) The Gaussian distribution centred on $t = 0$ with root mean square deviation $\Delta t = \tau$ is given by $f(t) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\tau^2}\right)$, $t \in (-\infty, \infty)$. If $\tilde{f}(\omega\}$ is the Fourier transformation of $f(t)$ with the Kernel $\frac{1}{\sqrt{2\pi}} e^{-i\omega t}$, show that $\Delta\omega\Delta t = 1$. 3

(c) If $z = x + iy$, show that the complex valued function $f(z) = |x| - i|y|$ is analytic, only in the second and fourth quadrants. 4

5. (a) For a spherically symmetric function $f(\mathbf{r}) = f(r)$ in three dimensions find the Fourier transform of $f(\mathbf{r})$ as a one-dimensional integral. 4

(b) A complex function $g(w)$ is define as 4

$$g(w) = \oint_C \frac{z^3 + 2z}{(z-w)^3}\, dz\,.$$

show that $g(w) = 6\pi i w$, when *w* is inside the contour *C*.

(c) Show that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are two similar matrices related through the non-singular matrix $\boldsymbol{S}$ as $\boldsymbol{B} = \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ have same set of eigenvalues. 2

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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