WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2022

MTMACOR08T-MATHEMATICS (CC8)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Find the lower and upper integrals of the function.

$$f(x)=\begin{cases}1 & ; \quad x\in\mathbb{Q}\\ 0 & ; \quad x\notin\mathbb{Q}\end{cases}$$

(b) Find the Cauchy Principal Value of $\int_{-1}^{1}\frac{dx}{x^5}$.

(c) Test the convergence of $\int_{0}^{2}\frac{\log x}{\sqrt{2-x}}\,dx$.

(d) Show that $B(m,\ n)=B(n,\ m)$, for $m, n>0$.

(e) Examine whether the sequence of functions $\{f_n\}$ converges uniformly on $\mathbb{R}$, where for all $n\in\mathbb{N}$,

$$f_n(x)=\frac{n}{x+n}\quad ,\quad x\in\mathbb{R}$$

(f) Find the limit function $f(x)$ of the sequence $\{f_n\}$ on $[0, \infty)$, where for all $n\in\mathbb{N}$,

$$f_n(x)=\frac{x^n}{1+x^n}\quad ,\quad x\geq 0$$

Hence, state with reason whether $\{f_n\}$ converges uniformly on $[0, \infty)$.

(g) Show that the series $\sum_{n=1}^{\infty}\frac{n^5+1}{n^7+3}\left(\frac{x}{2}\right)^n$ is uniformly convergent on [–2, 2].

(h) Find the radius of convergence of the power series: $\sum(-1)^{n-1}x^n$

2. (a) For bounded function f defined on an interval $[a, b]$ and any two partitions P_1, P_2 of $[a, b]$ show that $L(f,\ P_1)\ \leq\ U(f,\ P_2)$. 4

(b) Prove that a continuous function f defined on a closed interval $[a, b]$ is integrable in the sense of Riemann. 4

3. (a) A function $f:[0,1]\to\mathbb{R}$ is defined by 2+2

$$f(x)=\frac{1}{3^n}\quad,\quad \frac{1}{3^{n+1}}<x\le\frac{1}{3^n}\quad,\quad n=0,1,2,\ldots\ldots$$
$$=\;0\quad,\qquad x=0$$

Show that f is integrable in the sense of Riemann and $\int\limits_0^1 f(x)\,dx=\frac{3}{4}$.

(b) Using Mean Value Theorem of Integral Calculus prove that 4

$$\frac{\pi^3}{24}\le\int\limits_0^{\pi}\frac{x^2}{5+3\cos x}\,dx\le\frac{\pi^3}{6}$$

4. (a) Show that $\int\limits_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx=(b-a)^{m+n-1}\;\beta(m,n)\;,\;m,n>0$. 4

(b) Test the convergence of the integral $\int\limits_0^1\frac{\sqrt{x}}{e^{\sin x}-1}\,dx$. 4

5. (a) Let $f_n(x)=(x-[x])^n$, $x\in\mathbb{R}$, $n\in\mathbb{N}$. Show that the sequence $\{f_n\}$ is convergent pointwise. Verify whether the convergence is uniform. 2+2

(b) If $\{f_n\}$ is a sequence of functions defined on a set D converging uniformly to a function f on D such that each f_n is continuous at some point $c\in D$, prove that f is continuous at c. 4

6. (a) Verify the uniform convergence of the series 4

$$\sum_{n=0}^{\infty}\frac{x}{[(n+1)x+1][nx+1]}$$

on the interval $[a,b]$, where $0<a<b$.

(b) Show that the function $f(x)=\sum\limits_{n=1}^{\infty}\frac{\sin nx}{n^3}$ is differentiable on $\mathbb{R}$. Find its derivative. 4

7. (a) If a series $\sum\limits_{n=0}^{\infty}a_n x^n$ is convergent for some $x=a\ne 0$, then prove that the series converges absolutely for all x with $|x|<|a|$. 3

(b) Find the radius of convergence of the power series $\sum\limits_{n=1}^{\infty}\frac{x^n}{2^n n^2}$. Using this, show that the series $\sum\limits_{n=0}^{\infty}\frac{x^n}{2^{n+1}(n+1)}$ has the same radius of convergence. 3+2

8. (a) State Dirichlet's condition for convergence of a Fourier series. 2

   (b) Obtain the Fourier series expansion of $f(x)$ in $[-\pi,\ \pi]$ where 4+2

$$f(x)=\begin{cases} 0 & , \quad -\pi \le x < 0 \\ \frac{1}{4}\pi x & , \quad 0 \le x \le \pi \end{cases}$$

   Hence show that the sum of the series

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots=\frac{\pi^2}{8}$$

9. (a) The function $f:[-2,\ 2]\to \mathbb{R}$ is defined by 4

$$\begin{aligned} f(x) &= x+1 \quad , \quad -2 \le x \le 0 \\ &= x-1 \quad , \quad 0 < x \le 2 \end{aligned}$$

   Find the Fourier series of the function f.

   (b) Expand the function $f(x)=x^2$, $0 < x \le \pi$ in a Fourier Sine series. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within* 1 *hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

——×——