



WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 6th Semester Examination, 2022

MTMACOR13T-MATHEMATICS (CC13)

Time Allotted: 2 Hours

Full Marks: 50

*The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.*

Answer Question No. 1 and any *five* questions from the rest

1. Answer any *five* questions from the following: 2×5=10

(a) Let (X, d) be a metric space and $x_1, x_2, \dots, x_n \in X$. Prove that $d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-1}, x_n)$.

(b) Let (X, d) be a metric space. Prove that $\{x\}$ is a closed subset of X for all $x \in X$.

(c) Let $C[0, 1]$ be the set of all continuous real valued functions on the closed interval $[0, 1]$. Assume that d_1 and d_∞ are two metrics on $C[0, 1]$ where for all $f, g \in C[0, 1]$, $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$, $d_\infty(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$

Compute $d_1(f, g)$ and $d_\infty(f, g)$ for the functions

$$f(x) = x, \quad g(x) = x^2 \text{ for all } x \in C[0, 1].$$

(d) Show that the metric defined by $d(x, y) = |\tan^{-1} x - \tan^{-1} y|$ on $\mathbb{R}$ is a bounded metric.

(e) Show that $f(z) = \bar{z}$, $\forall z \in \mathbb{C}$ is a continuous function.

(f) At which point the function $f(z) = |z|^2 + i\bar{z} + 1$ is differentiable?

(g) If an analytic function $f(z)$ is such that $\operatorname{Re}\{f'(z)\} = 2y$ and $f(1+i) = 2$ then find the imaginary part of $f(z)$.

(h) Find the region of convergence of the series $\sum_{n=1}^{\infty} n! z^n$.

2. Let X be the set of all real sequences. Define $d : X \times X \rightarrow \mathbb{R}$ by 1+5+2

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \times \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

for all $x = \{x_n\}_n, y = \{y_n\}_n \in X$. Show that d is well defined. Prove that d is a metric on X . Is (X, d) a bounded metric space? Justify your answer.

3. (a) Let (X, d) be a metric space with $A \subset X$. Show that if $x \notin A$ and x is a limit point A then $d(x, A) = 0$. 4

(b) Define Cauchy sequence in a metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a metric space with $d(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$. If $\{x_n\}$ is a Cauchy sequence in X , prove that $\{y_n\}$ is also a Cauchy sequence. 4

4. (a) Prove that the space $C[a, b]$ of all continuous real-valued functions defined on $[a, b]$ with the metric $d(f, g) = \max_{t \in [a, b]} |f(t) - g(t)|$ is a complete metric space. 4

(b) Prove that a closed subset of a compact metric space (X, d) is compact. 4

5. (a) Let $f, g : (X, d_X) \rightarrow (Y, d_Y)$ be continuous functions. Prove that $\{x \in X : f(x) = g(x)\}$ is a closed subset of X . 4

(b) Let $f : (X, d_X) \rightarrow (Y, d_Y)$ be a continuous function where X is connected. Prove that $f(X)$ is a connected subset of Y . 4

6. (a) Show that the function 4

$$f(z) = \begin{cases} \frac{|z|}{\operatorname{Re} z}, & \operatorname{Re} z \neq 0 \\ 0, & \operatorname{Re} z = 0 \end{cases}$$

is not continuous at $z = 0$.

(b) Let $\omega \in \mathbb{C}$. Show that the function $f(z) = |z - \omega|^2$, $\forall z \in \mathbb{C}$ is differentiable only at ω . 4

7. (a) If $f(z) = \begin{cases} e^{-z^{-4}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ show that $f(z)$ is not analytic at $z = 0$ although C.R. equations are satisfied at the point $z = 0$. 4

(b) Let $f(z) = u(x, y) + i v(x, y)$ be differentiable at a point $z_0 = x_0 + y_0$. Then prove that the first order partial derivatives $u_x(x_0, y_0)$, $u_y(x_0, y_0)$, $v_x(x_0, y_0)$, $v_y(x_0, y_0)$ exist and they satisfy Cauchy-Riemann equations at a point (x_0, y_0) . Find $f'(z)$. 4

8. (a) Evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2} dz$ where C is the circle $|z| = 3$. 4

(b) Let f be an analytic function in a simply connected region D in the complex plane and let α, β be any two points in D . Prove that $\int_{\alpha}^{\beta} f(z) dz$ is independent of the path joining α and β . 4

9. (a) Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle $|z|=3$ 4
- (b) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + z^2}$ is convergent in the region $1 < |z| < 2$. 4

N.B. : Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.

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