WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 2nd Semester Examination, 2019

MTMACOR03T-MATHEMATICS (CC3)

REAL ANALYSIS

Time Allotted: 2 Hours — Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times 5=10$

(a) State Archimedean property of real numbers.

(b) Find the least upper bound and greatest lower bound of the set $T=\{x\in\mathbb{R}: 3x^2-10x+3<0\}$.

(c) Let G be an open set and F be a closed set of real numbers. Prove that $G-F$ is an open set.

(d) Show that the set of positive integers $\mathbb{N}$ is not compact by describing an open cover of $\mathbb{N}$ which has no finite subcover.

(e) Prove that $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$.

(f) Show that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a bounded sequence.

(g) Examine the convergence of the series $\sum\limits_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^2 x^{n-1}$ where $0<x<1$.

(h) Examine the convergence of the series $\sum\limits_{n=2}^{\infty}\frac{(-1)^{n+1}}{n\log n}$.

2. (a) Let A and B be two non empty bounded above subsets of $\mathbb{R}$. Let $C=\{x+y: x\in A,\ y\in B\}$. Show that $\sup C=\sup A+\sup B$. 3

(b) Let T be a bounded subset of $\mathbb{R}$. Define $S=\{|x-y|: x,\ y\in T\}$. Show that 5

$$\sup S=\sup T-\inf T$$

3. (a) Let A and B be two open sets of $\mathbb{R}$. Prove that $A\cap B$ is open in $\mathbb{R}$. Give an example with proper justification to show that arbitrary intersection of open subsets of $\mathbb{R}$ need not be open in $\mathbb{R}$. 2+2

(b) Let A and B be any two subsets of $\mathbb{R}$ such that $A\subseteq B$. Show that $A'\subseteq B'$, where X' is the derived set of X, for any $X\subseteq\mathbb{R}$. Give an example to show that an infinite subset of real numbers may not have an accumulation point in $\mathbb{R}$. 3+1

4. (a) Let A and B be two countably infinite subsets of $\mathbb{R}$. Show that $A \times B$ is a countably infinite subset of $\mathbb{R}$. 4

   (b) Prove that (0, 1) is an uncountable subset of $\mathbb{R}$. 4

5. (a) Prove that every compact subset of $\mathbb{R}$ is a bounded subset of $\mathbb{R}$. Is the converse true? Justify your answer. 2+2

   (b) Prove that every bounded closed interval of $\mathbb{R}$ is a compact subset of $\mathbb{R}$. 4

6. (a) Prove that every bounded sequence of real numbers has a convergent subsequence. 3

   (b) Let $\{x_n\}$ and $\{y_n\}$ be two bounded real sequences such that $\{x_n\}$ converges. Then prove that $\overline{\lim_{n\to\infty}}(x_n + y_n) = \lim_{n\to\infty} x_n + \overline{\lim_{n\to\infty}} y_n$. 3

   (c) Examine the convergence of the sequence $\{x_n\}$ where $x_n = \frac{n}{2} - [\frac{n}{2}]$ where $[\frac{n}{2}]$ has its usual meaning. 2

7. (a) If $\lim x_n = 0$ then prove that $\lim \dfrac{x_1 + x_2 + \cdots + x_n}{n} = 0$. 3+1

   Is the converse true? Give reasons.

   (b) Prove that the sequence $\left\{\dfrac{n}{n+1}\right\}$ is a Cauchy sequence. 2

   (c) Use Cauchy's general principle of convergence to prove that the sequence $\{u_n\}$ is not convergent where $u_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots\cdots + \dfrac{1}{n}$. 2

8. (a) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$. 3

   (b) Apply Cauchy's integral test to prove that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ $(p > 0)$ converges for $p > 1$ and diverges for $0 < p \le 1$. 3

   (c) Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$. 2

9. (a) State and prove Leibnitz's test for convergence of an alternating series. 1+3

   (b) Prove that an absolutely convergent series is convergent. 2

   (c) Give an example of a conditionally convergent series. Give reasons in support of your answer. 2

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